From Geometry to Algebra - Assessment 1

M7

22 October 2011

Question 1

Show that if ABC is a triangle whose circumscribed circle and inscribed circle have the same center, then ABC in an equilateral triangle.

A circumscribed triangle is one where a circle passes through each of the the points or corners of a triangle. A property of a circle is that the center of that circle is equidistant to any two, or more, points on that circle. If we have a circumscribed triangle, ABC, this means that the center of the circumscribed circle, O, is at an equal distance from the points A, B and C. Therefore the following can be said for all circumscribed triangles;

$$|AO| = |BO| = |CO|$$

Lemma 1. *The perpendicular bisector of a segment MN is the set of points equidistant to M and N. Where P is a point on the bisector, $|MP| = |PN|$.*

With the use of Lemma 1, mapping O to P, A to M and B to N, it can be said that the center of the circumscribed circle, O, lies on the perpendicular bisector of the line segment AB. By use of Lemma 1 it can also be said that the point O also lies on the perpendicular bisectors of AC and CB. Therefore the intersection point of the three perpendicular bisectors is the center of the circumscribed circle, O. This is shown below in Figure 1.

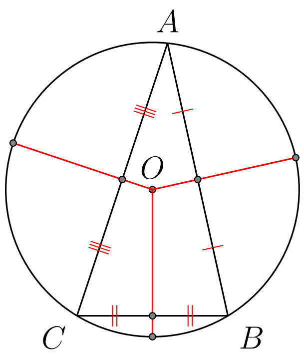

Figure 1: A triangle, ABC, with a circumscribed circle showing the perpendicular bisectors on each segment of the triangle.

An inscribed triangle is one where a circle passes at tangent to each of the triangles segments. A property of a circle is that the center of that circle is equidistant to any two, or more, points on that circle. If we have an inscribed triangle, ABC, this means that the center of the inscribed circle, O, is at an equal distance from the points D, E and F, where D, E and F are points on each of the segments of the triangle. A tangent is at $\frac{\pi}{2}$ to the radius of the circle. Therefore the following can be said for all inscribed triangles;

$$|DO| = |EO| = |FO| \qquad \text{where} \qquad \angle ADO = \angle BEO = \angle CFO = \frac{\pi}{2}$$

Lemma 2. *The bisector of an angle $\angle GHI$ is the division of that angle into two equal angles, $\angle GHJ$ and $\angle IHJ$ where J is any point on the angular bisector. It can be said that if $|GH| = |HI|$ then J is equidistant from both G and I.*

With the use of Lemma 2, mapping O to J, D to I and F to G, it can be said that the center of the inscribed circle, O lies on the angular bisector of $\angle CAB$. By use of Lemma 2 it can also be said that the point O also lies on the angular bisector of $\angle ABC$ and $\angle ACB$. Therefore the intersection point of the three angular bisectors is the center of the inscribed circle, O. This is shown below in Figure 2.

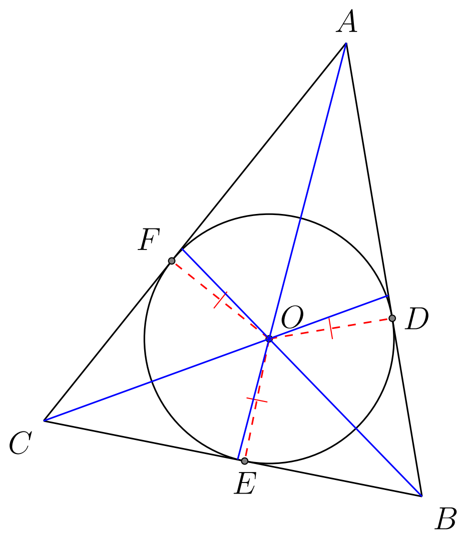

Figure 2: A triangle with an inscribed circle, center O, showing the segments DO, EO and FO (dashed), where D, E and F are points on the segments forming the triangle, at tangent to the circle, and the angular bisectors of $\angle CAB$, $\angle ABC$ and $\angle BCA$.

Now consider an isosceles triangle, where $|OB| = |OC|$.

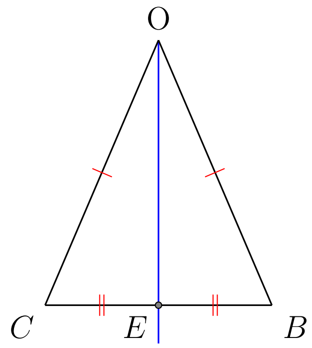

Figure 3: An isosceles triangle.

As OBC is an isosceles triangle, $\angle OBC = \angle OCB$. If we assume E to be a point on segment BC such that it is the intersection of the angular bisector of $\angle BOC$ and segment BC, we can show that E is the mid-point of BC and that the angular bisector of $\angle BOC$ is also the perpendicular bisector of segment BC.

Firstly we can show that $OEC \cong OEB$ by SAS as $|OE|$ is a shared side, $\angle OBC = \angle OCB$ and $|OB| = |OC|$.

This implies that $|CE| = |EB|$ and $|BC| = |CE| + |BE|$ therefore $|BC| = 2|BE|$. This also implies that $\angle OEC = \angle OEB$, as they are on a straight line they equal π so $\angle OEC$ and $\angle OEB$ are equal to $\frac{\pi}{2}$ and thus the parallel bisector.

This can be generalized into the following statements.

Lemma 3. *For any isosceles triangle, the angular bisector of the angle between the two sides of equal length coincide with the perpendicular bisector of the third side.*

Lemma 4. *For any isosceles triangle, the angular bisector that coincides with the perpendicular bisector forms a divide in that triangle to form two new congruent triangles.*

Now lets construct a triangle, ABC where the circumscribed circle and inscribed circle have the same center, O. The following can be said for this triangle;

1. From the circumscribed circle, $|AO| = |BO| = |CO|$,
2. From the inscribed circle,

$$|DO| = |EO| = |FO| \quad \text{where} \quad \angle ADO = \angle BEO = \angle CFO = \frac{\pi}{2}$$

Using the statements; $|BO| = |CO|$ and $\angle BEO = \frac{\pi}{2}$ we can say that BOC is an isosceles triangle and that $|BE| = |EC|$ and therefore $|BC| = 2|BE|$.

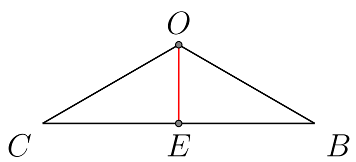

Figure 4: BOC.

Using the statements; $|AO| = |BO|$ and $\angle ADO = \frac{\pi}{2}$ we can say that AOB is an isosceles triangle and that $|AD| = |DB|$ and therefore $|AB| = 2|DB|$.

Using the statements; $|AO| = |CO|$ and $\angle CFO = \frac{\pi}{2}$ we can say that AOC is an isosceles triangle and that $|AF| = |FC|$ and therefore $|AC| = 2|FC|$.

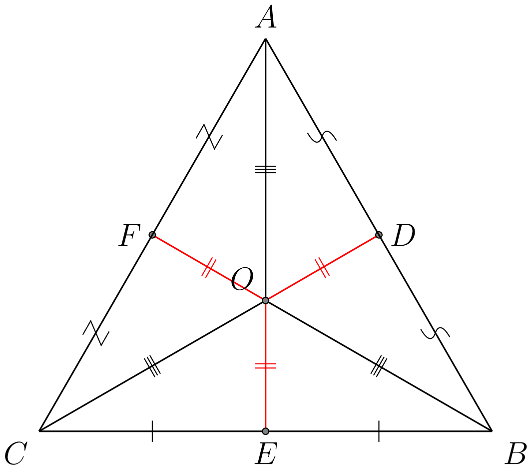

Figure 5: Triangle ABC.

Figure 5 shows triangle ABC in full. As $|OE| = |OD|$, $|OB|$ is a shared side and $\angle OEB$ and $\angle ODB$ are $\frac{\pi}{2}$, by Side,Side,$\frac{\pi}{2}$ we can say that they are congruent. $OEB \cong ODB$.

As $|OE| = |OF|$, $|OC|$ is a shared side and $\angle OEC$ and $\angle OFC$ are $\frac{\pi}{2}$, by Side,Side $\frac{\pi}{2}$ we can say that they are congruent. $OEC \cong OFC$.

As $|OF| = |OD|$, $|OA|$ is a shared side and $\angle OFA$ and $\angle ODA$ are $\frac{\pi}{2}$, by Side,Side $\frac{\pi}{2}$ we can say that they are congruent. $OFA \cong ODA$.

By the use of Lemma 4 we can also say;

$$ODA \cong ODB, OEB \cong OEC, OFC \cong OFA.$$

Therefore;

$$ODA \cong ODB \cong OEB \cong OEC \cong OFC \cong OFA.$$

This implies that;

$$|AD| = |DB| = |BE| = |EC| = |CF| = |FA|.$$

As $|AB| = 2|DB|$, $|BC| = 2|BE|$ and $|AC| = 2|FC|$ it can be said that;

$$|AB| = |BC| = |AC|$$

and therefore ABC is an equilateral triangle.

It can be noted that if a triangle has an inscribed circle and circumscribed circle with the same center, then in must be an equilateral triangle. Equilateral triangles have equal angles, each being $\frac{\pi}{3}$.

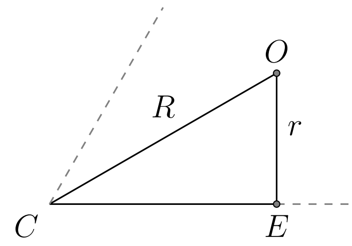

Figure 6: Relationship of R to r.

As O, the center of the circles, lies at a point on the angular bisector and the angle between a side of the triangle and the angular bisector, $\angle OCE$, is $\frac{\pi}{6}$, the following relationship exists between R, the radius of the circumscribed circle, and r, the radius of the inscribed circle.

$$R \sin(\frac{\pi}{6}) = r \iff r = \frac{R}{2}$$

Question 2

Suppose that ℓ_1, ℓ_2 and ℓ_3 are three distinct lines intersecting each other in a point P as in the Figure below. Let r_i be the reflection in ℓ_i for $i = 1, 2, 3$. Which points stay fixed under $r_3 \circ r_2 \circ r_1$? What is $r_3 \circ r_2 \circ r_1$?

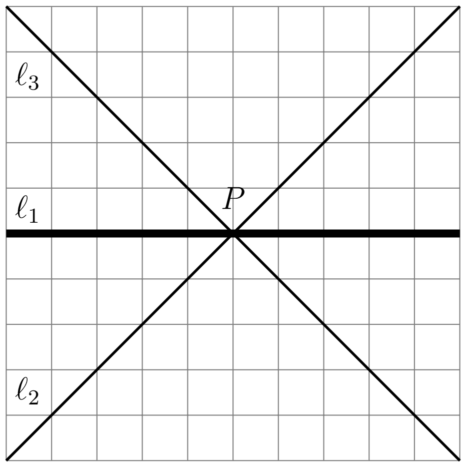

Figure 7: Question 2

Lemma 5. *All points on a line, remain in the same place on that line upon reflection of that line.*

With the use of Lemma 5, we can say that if a point lies of the intersection of multiple lines then it will stay in the same place upon reflection. Therefore Point P in Figure 7 will stay in the same place upon reflection of $r_3 \circ r_2 \circ r_1$.

Lets now construct three points on this map and transform the map by $r_3 \circ r_2 \circ r_1$.

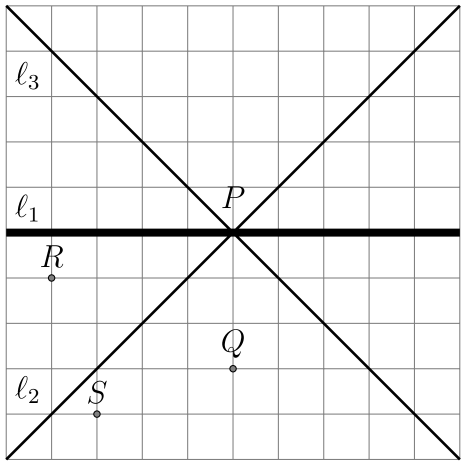

Figure 8: Initial points

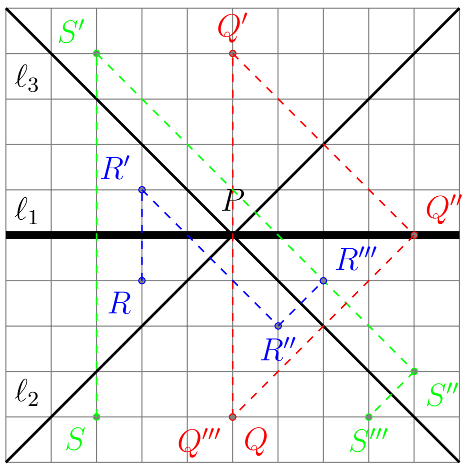

Figure 9: Reflected points

It is shown in Figure 9 that Q''' maps onto Q under $r_3 \circ r_2 \circ r_1$ and that R and S have mapped to R''' and S''' by translation in the horizontal, P remains at the same point as it lies on all of the lines of the reflection. By looking closer at these points, it can be noted that P and Q have not moved under $r_3 \circ r_2 \circ r_1$. If we assume that these points P and Q line on another line, ℓ_4, then under reflection upon that line they would not move according to Lemma 5. If this assumption is true then the line ℓ_4 is perpendicular bisector of RR''' and SS'''. This is the case in Figure 9.

Therefore the following can be said, where r_4 is the reflection in the line ℓ_4;

$$r_3 \circ r_2 \circ r_1 = r_4$$

This implies with the use of Lemma 5 that any point on ℓ_4 will also stay fixed under $r_3 \circ r_2 \circ r_1$

This is also the case if the lines are changed, but also intersect at point P.

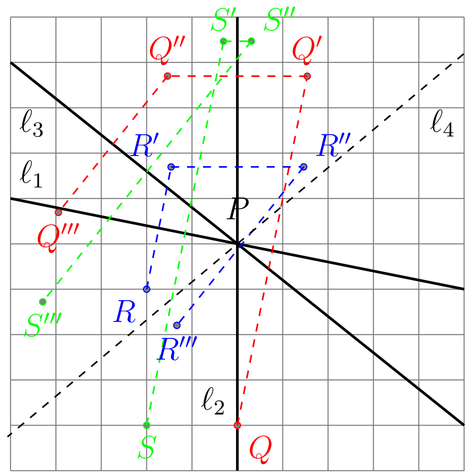

Figure 10: Three different lines

Figure 10 shows three lines intersecting at point P, that are different from Figure 7. It is also the case that a line, ℓ_4 can be constructed so that ℓ_4 is a line on the mid point of all segments XX''' for each point, X, on the map. In this case;

$$r_3 \circ r_2 \circ r_1 = r_4$$

Where r_4 is the reflection in the line ℓ_4.

The following statement can be made;

If three lines in an euclidean place meet at a single point then they are concurrent. Three reflections in concurrent lines can be reduced to one reflection. [1]

References

[1] Calkins, Keith G, *A Review of Basic Geometry*. [Online] Mirror, Mirror... Available from: http://www.andrews.edu/∼calkins/math/webtexts/geom04.htm [Accessed 2 November 2011]